ME 121: Mass Additions to Control Fish Tank Salinity Gerald Recktenwald
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1 Goal: Use Mass Balances to Predict Control Corrections

Our goal is to keep the salinity of the water in the fish tank within a relatively narrow range of values. If the water becomes too salty, the control algorithm signals to open a solenoid that adds fresh water. If the water is not salty enough, the control algorithm signals to open the solenoid that adds salty water. A good control algorithm will open the appropriate valve for an amount of time that depends on how far the system is out of balance.

Whether the salinity is too high or too low, the size of the correction should be large enough to bring the salinity back into balance, or at least make a significant change in salinity in the desired direction. On the other hand, the size of the correction should not be so large that the salinity overshoots. For example, if the salinity is initially too low, the control algorithm should not open the solenoid from the salty supply so long that the salinity in the tank becomes too high.

In these notes we develop a model (a set of equation) to predict the size of the needed correction given a desired setpoint and a salinity reading that is beyond the deadband around the setpoint. The model assumes that the size of the correction is proportional to the difference between the current salinity reading and the setpoint. The starting point for the model is a mass balance for the salt and water components in the fish tank. The model allows for mass to enter the fish tank from the reservoirs of salty and fresh water, and to leave the fish tank from through the overflow.

2 Using Gain to Set the Size of the Correction

The setpoint, X_{sp}, is the desired mass fraction of salt in the tank. The deadband is a range of salinities that are close enough to the setpoint that no correction needs to be made. When the measured salinity deviates outside the deadband around X_{sp}, the control system adds salty or fresh water to bring the salinity back to within the deadband.

Suppose that a disturbance causes the salinity to equal $X_{s,i}$, where the “i” indicates *initial.* In this context, initial designates the state of the system *before* a correction is made. After a correction, the mass fraction is $X_{s,f}$. Refer to Table 1 for a list of terms.

Table 1: Nomenclature for the salinity control model.

E	Error in the salinity relative to the setpoint, $E = X_{s,i} - X_{\text{sp}}$.
G	Gain, a dimensionless fraction used to specify the amount of correction to make for one cycle through the control loop.
X_{sp}	Mass fraction of salt at the set point.
$X_{s,i}$	Initial mass fraction, which is the value of the mass fraction of salt *after* a disturbance to the tank salinity has occurred.
$X_{s,f}$	Final mass fraction of salt after a correction has been made. We *do not* assume that $X_{s,f} = X_{\text{sp}}$.

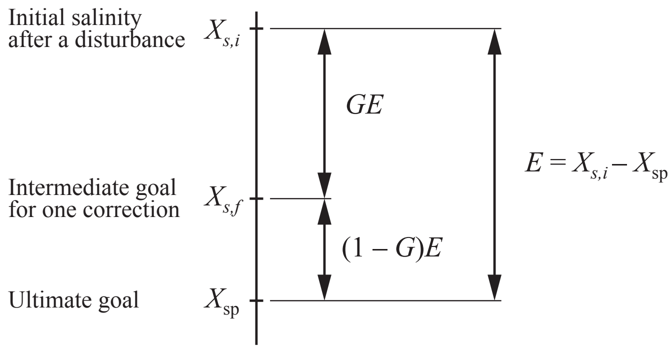

Figure 1: Relationship between gain, G and the size of the correction made to the salinity. In this diagram, the initial state of the fish tank is assumed to be too salty, i.e., $X_{s,i} > X_{\text{sp}}$. The target salinity, $X_{s,f}$, is less than $X_{s,i}$, meaning that the correction reduces the salinity, but not all the way to X_{sp}. A similar diagram could be created for the case where $X_{s,i} < X_{\text{sp}}$. The algebra still works.

For a given X_{sp} and $X_{s,i}$ the error, E is defined as the deviation between the current salinity mass fraction and the desired set point.

$$E = X_{s,i} - X_{\text{sp}} \tag{1}$$

The error is positive when the fish tank is too salty, and negative when it is not salty enough.

The gain, G, is a fraction (usually less than 1.0) that determines how much of a correction to make to the system. We use the gain to avoid an overshoot, i.e., from making a correction that is too large.

Figure 1 graphically represents the relationship between E, G, $X_{s,i}$, $X_{s,f}$and X_{sp}. During operation of the control system, the user prescribes X_{sp} and G. The value of X_s is measured continuously. When $X_{s,i}$ is outside the deadband, a target value for $X_{s,f}$ is computed from

$$\begin{aligned} X_{s,f} &= X_{\text{sp}} + (1 - G)E \\ &= X_{\text{sp}} + (1 - G)(X_{s,i} - X_{\text{sp}}) \end{aligned} \tag{2}$$

Equation (2) applies whether $X_{s,i} > X_{\text{sp}}$ or $X_{s,i} < X_{\text{sp}}$. Also note that Equation (2) is simply a user-specified target for a single step in the control algorithm. Equation (2) is not derived from, or dependent upon, a mass balance. Additional information about the system is needed in order to determine how much mass needs to be added to achieve $X_{s,f}$, i.e., to reach the target, for a given $X_{s,i}$, X_{sp} and G.

3 A Model for the Correction

Corrections to the system salinity are made by adding water from either the salty or fresh reservoirs. Figure 2 is a batch model of the correction during one step of the control algorithm.

A mass balance for the mixture (water plus salt) during the batch process depicted in Figure 2 is

$$m_{t,i} + m_{\text{add}} = m_{t,f} + m_{s,\text{over}} \tag{3}$$

where $m_{t,i}$ is the initial mass of mixture in the tank, m_{add} is the mass added from one of the reservoirs, $m_{t,f}$ is the final mass (after the correction) of mixture in the tank, and $m_{s,\text{over}}$ is the mass of mixture that leaves the tank through the overflow.

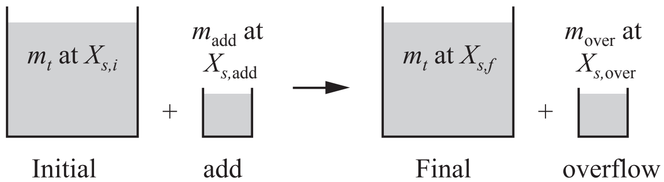

Figure 2: The batch model for mass addition and overflow. The initial state (to the left of the arrow) consists of the water in the fish tank before the correction, and the mass (fresh or salty) to be added. The final state (to the right of the arrow) consists of the water in the fish tank after the correction, and the mass of water that leaves through the overflow.

3.1 A Simplifying Assumption

The algebra of the correction model can be simplified if we *assume* that

$$m_{t,i} = m_{t,f}. \tag{4}$$

This assumption can be justified with the following observations.

- In steady state, the overflow for the tank keeps the volume of water in the tank relatively constant.
- Overall mass fractions of salt are very low, which means that the mass of mixture in the tank is overwhelmingly due to the mass of the water.
- Since the mass fractions of salt are low, the density of the fluid in the tank will not change significantly within the expected range of salinities.
- If the volume of fluid in the tank before and after the correction is the same, and if the density of the fluid changes very little during operation of the tank, then $m_{t,f} \approx m_{t,i}$.

Note that any error caused by the assumption of Equation (4) will only effect the accuracy of the prediction for the amount of water to add from one of the reservoirs. With $G < 1$, the correction during one control step will is *designed to* bring the system close to, but not fully into equilibrium at X_{sp}. Therefore, multiple corrections are likely to be necessary, and a small error in any one correction should not cause trouble. If the system does experience overshoots, reducing the value of G should help.

Using the assumption of Equation (4) in Equation (3), we obtain the following convenient result

$$m_{\text{add}} = m_{s,\text{over}}. \tag{5}$$

In other words, since the tank is full and any addition of water results in overflow to the drain, the mass of water leaving the tank through the overflow is equal to the mass of water added from one of the reservoirs.

Note that we could have started with the assumption that Equation (5) is true, which, when combined with Equation (3) gives Equation (4). However, to justify starting with Equation (5), we would still need to make assumptions like those listed in the bullets under Equation (4).

3.2 Short-circuiting at the Outflow

A correction by the control system causes a solenoid valve attached to a reservoir to open, which allows water from the reservoir to flow into the fish tank. The tank is relatively small, and the overflow port and the outlets of the supply tubes from the reservoirs are near the top of the tank. Therefore, it is reasonable to assume that the water in the tank is not fully mixed during an addition of water from the supply reservoirs. As a consequence, some of the water from the reservoir is likely to be short-circuited directly to the outflow port.

It is difficult to know how much of m_{add} flows directly into the outlet and how much mixes with the water in the tank. We will use a simple model to estimate the effect of the short-circuiting of reservoir water. Suppose that the salinity of water leaving the overflow port can be described by a parameter F, which is the fraction of the overflow water that comes from the supply reservoir. The remaining fraction overflow water comes from the tank. Thus, the mass fraction of salt in the overflow mixture is

$$X_{s,\text{over}} = F X_{s,\text{add}} + (1-F) X_{s,i} \tag{6}$$

where $0 \leq F \leq 1$, $X_{s,\text{add}}$ is the salinity of water being added from the reservoir, and $X_{s,i}$ (as before) is the salinity in the tank before the correction is made.

3.3 Computing the Mass to Add During Correction

We now have enough pieces to predict how much water must be added from the reservoir in order to make the desired correction. Begin by writing the mass balance for the salt for the batch process depicted in Figure 2,

$$m_{\text{add}} X_{s,\text{add}} + m_{t,i} X_{s,i} = m_{t,f} X_{s,f} + m_{s,\text{over}} X_{s,\text{over}} \tag{7}$$

Introducing the simplifications from Equation (4) and Equation (5) gives

$$m_{\text{add}} X_{s,\text{add}} + m_{t,i} X_{s,i} = m_{t,i} X_{s,f} + m_{\text{add}} X_{s,\text{over}} \tag{8}$$

Collecting terms and solving for m_{add} yields

$$m_{\text{add}} = m_{t,i} \, \frac{X_{s,i} - X_{s,f}}{X_{s,\text{over}} - X_{s,\text{add}}} \tag{9}$$

Using Equation (6) to substitute for $X_{s,\text{over}}$ gives

$$m_{\text{add}} = m_{t,i} \, \frac{X_{s,i} - X_{s,f}}{F X_{s,\text{add}} + (1-F) X_{s,i} - X_{s,\text{add}}} \; = \; m_{t,i} \, \frac{X_{s,i} - X_{s,f}}{(1-F) X_{s,i} - (1-F) X_{s,\text{add}}}$$

Making some final rearrangements to the preceding equation gives.

$$\boxed{m_{\text{add}} = m_{t,i} \left(\frac{1}{1-F} \right) \frac{X_{s,i} - X_{s,f}}{X_{s,i} - X_{s,\text{add}}}} \tag{10}$$

4 Algorithm for Mass Addition

The equations in the preceding sections are used in the control algorithm for correcting the salinity in the fish tank. To operate the fish tank control, we first choose the setpoint, X_{sp}, and the control parameters F and G, which are assumed to be constant while the tank is operating. Then, to keep the fish tank salinity within the deadband around X_{sp}, perform the following steps.

1. Take a reading from the salinity sensor.
2. If the salinity reading is outside of the deadband around X_{sp}, perform the following steps. Otherwise, return to step 2.
   a. Compute the *target value* of $X_{s,f}$ from Equation (2).
   b. Compute the *mass addition* from Equation (10).
   c. Open the appropriate solenoid valve for a duration of time to allow m_{add} to flow into the tank. See Equation (12) in Section 5.1, below.

Note that choice of valve and the corresponding value of $X_{s,\text{add}}$ in step (2c) is determined by whether the salinity is too high or too low.

4.1 Example Calculation of Mass Additions

In this section, the model equations for the control algorithm are used to predict the required mass addition for two cases when the salinity error is positive and when it is negative.

Example 1 Correct a Salty Tank

How much DI water needs to be added given the following values

$$X_{\text{sp}} = 0.11\%, \qquad X_{s,i} = 0.14\%, \qquad G = 0.8$$

when the supply tanks have the following mass fractions of salt?

$$X_{\text{salty}} = 1\%, \qquad X_{\text{fresh}} = 0\%$$

Begin by computing that target value of tank salinity after the correction from Equation (2)

$$X_{s,f} = X_{\text{sp}} + (1-G)(X_{s,i} - X_{\text{sp}}) = 0.11 + (0.2)(0.14 - 0.11) = 0.116$$

Now compute the amount of water to add from Equation (10). Since we have no information about the size of the tank, we will compute the ratio $m_{\text{add}}/m_{t,i}$

$$\frac{m_{\text{add}}}{m_{t,i}} = \left(\frac{1}{1-F}\right)\frac{X_{s,i} - X_{s,f}}{X_{s,i} - X_{s,\text{add}}} \tag{11}$$

We need to make an assumption about F. We will choose $F = 0.15$, which means that 15% of the fluid from the supply reservoir is short-circuited to the drain. Since $X_{s,i} > X_{\text{sp}}$ fresh water needs to be added to correct the salinity. Therefore, $X_{s,\text{add}} = 0$. Substituting numerical values gives

$$\frac{m_{\text{add}}}{m_{t,i}} = \left(\frac{1}{0.85}\right)\frac{0.14 - 0.116}{0.14 - 0} = 0.202$$

Therefore, changing the salinity from $X_{s,i} = 0.14$ to $X_{s,f} = 0.116$ requires adding an amount of DI water equal to roughly 20 percent of the tank volume. □

Example 2 Correct a Tank that is not Salty Enough

How much salty water needs to be added given the following values

$$X_{\text{sp}} = 0.11\%, \qquad X_{s,i} = 0.08\%, \qquad G = 0.8$$

when the supply tanks have the following mass fractions of salt?

$$X_{\text{salty}} = 1\%, \qquad X_{\text{fresh}} = 0\%$$

The analysis procedes as in Example 1. First compute the target mass fraction.

$$X_{s,f} = X_{\text{sp}} + (1 - G)(X_{s,i} - X_{\text{sp}}) = 0.11 + (0.2)(0.08 - 0.11) = 0.104$$

Since $X_{s,i} < X_{\text{sp}}$, water from the salty reservoir determines $X_{s,\text{add}}$.

$$\frac{m_{\text{add}}}{m_{t,i}} = \left(\frac{1}{0.85}\right)\frac{0.08 - 0.104}{0.08 - 1.0} = 0.031$$

For the initial conditions, an addition of 3 percent of the tank mass is sufficient to shift the tank Salinity. □

Notice that in both Example 1 and Example 2, the size of the error ($|E| = |X_{s,i} - X_{s,f}|$) is the same. However, the fractional mass correction is much larger for the initially salty tank than it is for the initially unsalty tank. The difference is due to the difference in concentration of the supply reservoirs. For the initially salty tank in Example 1, the difference between the salt mass fraction in the supply reservoir and $X_{s,i}$ is much smaller than that difference for the tank that is initially not salty enough.

5 Modifications to the Basic Mass Correction Algorithm

The algorithm in Section 4 needs additional work before we can use it to control salinity of the fish tank. First, we need a separate model for the relationship between m_{add} and the amount of time to leave the valve open. Second, we need to allow for the water in the tank to mix before making additional corrections.

5.1 Estimate Δt_{add} from Valve Flow Rate

The Arduino microcontroller running the control algorithm can only open and close valves in response to salinity measurement. It cannot directly control the addition of water from either the salty or fresh water reservoirs. Instead, the control algorithm determines *how long* either of the valves is open. Therefore, we need a model for the amount mass of water enterring the tank while the valve is open. We will need to perform a simple calibration experiment to use this model.

Suppose that when a solenoid valve is open, the mass flow rate is steady and equal to $\dot{m}_{\text{sol}}$. If the valve is open for a period of time Δt_{add}, then the amount of mass added is

$$m_{\text{add}} = \dot{m}_{\text{sol}} \Delta t_{\text{add}}$$

Solving for Δt_{add} gives the time that the solenoid should be open to achieve a desired mass addition,

$$\Delta t_{\text{add}} = \frac{m_{\text{add}}}{\dot{m}_{\text{sol}}} \tag{12}$$

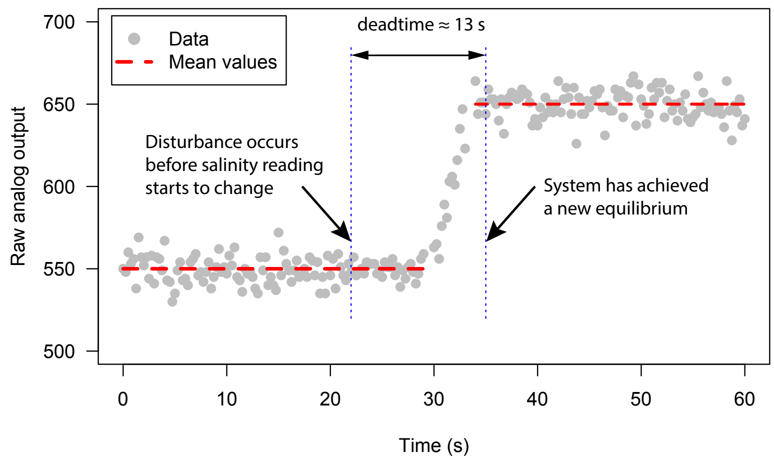

Figure 3: Illustration of the deadtime between making an adjustment and reaching a new equilibrium.

where $\dot{m}_{\mathrm{sol}}$ is the mass flow rate through the valve when it is open. Equation (12) is used in step (2c) of the algorithm in Section 4.

Equation (12) is based on the assumption that the mass flow rate is instantaneous when the valve is opened, and that the mass flow rate is steady during the time that the valve is open. Although neither of those assumptions are true, we use Equation (12) anyway.

The value of $\dot{m}_{\mathrm{sol}}$ is obtained from calibration experiments on the solenoid valves[1]. The calibration results are obtained by measuring how much water flows from the solenoid when it is open for known amount of time, say one second. During normal operation of the control algorithm, neither valve is likely to be open for one second or more. Because the water in the system has some inertia, and because of flow resistance in the valve and tubing, the flow rate for short bursts of valve openings is likely to be less than the flow rate during longer valve openings. Therefore, the value of $\dot{m}_{\mathrm{sol}}$ obtained during the calibration is likely to be higher than the actual $\dot{m}_{\mathrm{sol}}$ achieved when the valve is open for a shorter period of time.

Despite the simplicity of the flow rate model, Equation (12) is still a useful model and it gives a conservative estimate of Δt_{add}. In this context, *conservative estimate* means that during Δt_{add} the amount of mass actually added will be a less than what is necessary to fully correct the salinity error. Therefore, like $G < 1$, the assumptions leading to Equation (12) will help to prevent overshoot.

5.2 Deadtime Between Corrections

After fresh or salty water is added to the fish tank, some time must pass before the water in the fish tank system – the tank, the tubing, the pump, and the salinity sensor – is fully mixed. To make sure the control algorithm does not overreact, we need to wait long enough after one correction for the water in the system to mix before making another correction.

Figure 3 illustrates the time lag between when a salinity correction is made (by mass addition), and the establishment of a new salinity equilibrium. The data in Figure 3 is idealized: measurements on your fish tank may not look as clean. In general the salinity readings will fluctuate around some mean value, as indicated by the dashed red lines in Figure 3. The time between the initial disturbance and the establishment of the new equilibrium is called the *deadtime*[2].

[1] The calibration experiments are described in another document.

[2] Do not confuse *deadband* and *deadtime*.

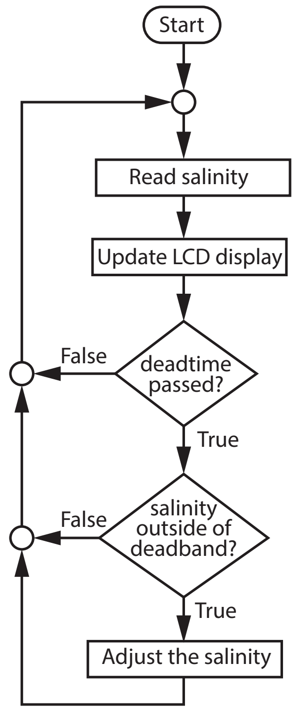

Figure 4: Flowchart for decision on whether and when to adjust the salinity.

The deadtime is the time necessary for the system to fully mix after a salinity correction by mass addition. The deadtime depends on the volume of water in the system, the flow rate produced by the pump, and (to a lesser extent) on the efficiency of mixing in the tank volume. The value of deadtime for your fish tank is determined by yet another calibration experiment described in another document.

5.3 Flow Chart for Salinity Control

Figure 4 is the overall flow chart for the salinity control algorithm. The deadtime determines *when* it is OK to adjust the salinity, not how it is adjusted. If the time since the last adjustment is less than the deadtime, no correction is made regardless of whether the measured salinity is outside of the deadband. In fact, if the time since the last adjustment is less than the deadtime, the control algorithm does not even bother to determine whether a correction should be made. The last code block in Figure 4 is the algorithm in Section 4.

6 Summary

This article describes how to compute the size of the mass correction needed to bring the salinity of the fish tank back to the setpoint. The use of the gain ($G < 1$) and the assumptions about the flow rate through the solenoid valves result in mass additions that will not fully bring the tank back to the setpoint in one correction. The smaller mass additions are desirable since they will help prevent the system from overshooting the setpoint.

Section 4 gives the algorithm for returning to equilibrium when the salinity is outside of the deadband. Figure 4 shows the logic for deciding *when* the algorithm for computing the size of the correction is used.